

THE TRACE PROBLEM FOR VECTOR FIELDS SATISFYING HÖRMANDER'S CONDITION.

S. BERHANU AND I. PESENSEN

ABSTRACT. Trace theorems are proved for non-isotropic Sobolev and L^p -Lipschitz spaces defined by vector fields satisfying Hörmander's bracket condition of order 2. It is shown that the loss of regularity by traces is the same as in the classical case.

0. INTRODUCTION.

It is a classical fact that for Sobolev spaces $W_p^r(\mathbb{R}^{n+1})$, the space of traces is the L^p -Lipschitz (Besov) space $\Lambda_p^{r-1/p}(\mathbb{R}^k)$. This result when $p = 2$ was obtained in [1] and [9], and for $r = 1$ and $1 < p < \infty$ in [3]. The complete solution for the integer and fractional Sobolev spaces was obtained by E. Stein [10] and for the Besov spaces by O. Besov [2].

In this paper we consider the analogous problem for non-isotropic Sobolev and L^p -Lipschitz spaces. It turns out that as in the classical case, the space of all traces can be described in terms of some kind of Besov norm constructed by means of “tangential components” of the given vector fields and their one-parameter groups of diffeomorphisms. We have the same phenomenon as in the classical situation: traces are less regular than the original functions and the loss of regularity is precisely $\frac{1}{p}$. We prove both restriction and extension theorems that are compatible. The extension result is established by analyzing an explicit extension operator which is a non-isotropic version of the classical Hardy operator. In the model case on the Heisenberg group, this problem was solved in [5] and [6].

This article is organized as follows. In section 1 we first state our results and prove the independence of our function spaces on the bases used. We then present the proof of our restriction result, Theorem 1.1. In section 2 we present our extension operator and prove the extension theorem, Theorem 1.2.

1. STATEMENT OF RESULTS AND INDEPENDENCE OF BASES.

For a point in \mathbb{R}^{n+1} , we will use coordinates (x, t) where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ and view \mathbb{R}^{n+1} as $\mathbb{R}_x^n \times \mathbb{R}_t$. We will also identify the subset $\mathbb{R}_x^n \times \{0\}$ with \mathbb{R}_x^n . For any $y \in \mathbb{R}^{n+1}$, the vector space $T_y \mathbb{R}_x^n$ will denote

$$\left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} : a_j \in \mathbb{R} \text{ for } j = 1, \dots, n \right\}.$$

Let \mathcal{V} be a C^∞ vector subbundle of the tangent space $T\mathbb{R}^{n+1}$ near 0. Let the fiber dimension of \mathcal{V} be $k+1$. For any point y where \mathcal{V} is defined, \mathcal{V}_y will denote the fiber of \mathcal{V} at y . We will assume that \mathcal{V} satisfies the following two conditions:

- (i) $\mathcal{V}_0 \subsetneq T_0\mathbb{R}_x^n$, and
- (ii) The sections of $\mathcal{V} \cap T\mathbb{R}_x^n$ together with their brackets $[X, Y]$ span $T\mathbb{R}_x^n$ near 0 in \mathbb{R}_x^n .

Assumption (i) means that there is a vector v in the fiber \mathcal{V}_0 with a nonzero $\frac{\partial}{\partial t}$ component. It follows that $\mathcal{V} \cap T\mathbb{R}_x^n$ forms a bundle of fiber dimension k near 0. Indeed, since $\mathcal{V}_0 + T_0\mathbb{R}_x^n = T_0\mathbb{R}^{n+1}$, by continuity, $\mathcal{V}_y + T_y\mathbb{R}_x^n = T_y\mathbb{R}^{n+1}$ for y near 0. Hence $\mathcal{V}_y \cap T_y\mathbb{R}_x^n$ is of dimension k for y near 0, telling us that $\mathcal{V} \cap T\mathbb{R}_x^n$ is a bundle. Condition (ii) therefore says that the restriction of this bundle to $\mathbb{R}_x^n \times \{0\}$ satisfies Hörmander's bracket condition of order 2.

Here is a simple example in $\mathbb{R}^4 = \mathbb{R}_x^3 \times \mathbb{R}_t$, where $x = (x_1, x_2, x_3)$. Let \mathcal{V}' be the C^∞ bundle generated by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial t}$, and $\frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$. Then $\frac{\partial}{\partial t} \notin T_0\mathbb{R}_x^3$ and so (i) is met. Since $\mathcal{V}' \cap T\mathbb{R}_x^3$ is generated by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$, and $[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}] = \frac{\partial}{\partial x_3}$, we see that (ii) is also met.

Let $\beta = \{Z_1, \dots, Z_k\}$ be a basis of $\mathcal{V} \cap T\mathbb{R}_x^n$ over an open neighborhood V of 0 in \mathbb{R}_x^n . Let V_1 be a neighborhood of 0 such that $V_1 \subset\subset V$ and suppose $\delta > 0$ satisfies

$$e^{\tau Z_j}(V_1) \subseteq V \text{ for } |\tau| \leq \delta \text{ and for all } j.$$

(Here $e^{\tau Z_j}x$ denotes the integral curve of Z_j starting at x when $\tau = 0$). Let $1 < p < \infty$. For $\psi \in C_0^\infty(V_1)$, define

$$\omega(t, \psi, Z_j, V_1, V) = \sup_{|\tau| \leq t} \|e^{\tau Z_j} \psi - \psi\|_{L^p}$$

and

$$\|\psi\|_{W_{1-\frac{1}{p}, p}(\beta, V, V_1, \delta)} = \|\psi\|_{L^p} + \sum_{i=1}^k \left\{ \int_0^\delta [t^{-\theta} \omega_i(t, \psi, V)]^p dt \bar{e} \tau t \right\}^{\frac{1}{p}},$$

where $\theta = 1 - \frac{1}{p}$ and $\omega_i(t, \psi, V) = \omega(t, \psi, Z_i, V_1, V)$.

Note that if $0 < \delta' < \delta$, then

$$\|\psi\|_{W_{1-\frac{1}{p}, p}(\beta, V, V_1, \delta)}$$

is equivalent to

$$\|\psi\|_{W_{1-\frac{1}{p}, p}(\beta, V, V_1, \delta')}$$

and hence, in the sequel, we'll simply write

$$\|\psi\|_{W_{1-\frac{1}{p}, p}(\beta, V, V_1)}$$

with the implicit understanding that we are using some $\delta > 0$ satisfying

$$e^{\tau Z_j}(V_1) \subseteq V \text{ for } |\tau| \leq \delta \text{ and } j = 1, \dots, k.$$

We will next show that if we change the basis β to β' , then after contracting the neighborhoods V and V_1 , the norms become equivalent. More precisely, we have:

Lemma 1.1. *Let $\beta = \{Z_1, \dots, Z_k\}$ and $\beta' = \{Y_1, \dots, Y_k\}$ be bases of $\mathcal{V} \cap T\mathbb{R}_x^n$ over a neighborhood V of 0 in \mathbb{R}_x^n . Then there exist neighborhoods $V_2 \subset \subset V_1 \subset \subset V$ and $C > 0$ such that for all $\psi \in C_0^\infty(V_2)$,*

$$\|\psi\|_{W_{1-\frac{1}{p},p}(\beta', V_2, V_1)} \leq C \|\psi\|_{W_{1-\frac{1}{p},p}(\beta, V_2, V_1)}.$$

Proof. Since the Z_j together with their brackets span $T\mathbb{R}_x^n$ near 0, after contracting V if necessary, we get a basis

$$\tilde{\beta} = \{Z_1, \dots, Z_k, Z_{k+1}, \dots, Z_n\}$$

of $T\mathbb{R}_x^n$ over V where for each $i \geq k+1$, $Z_i = [Z_1^i, Z_2^i]$ for some Z_1^i, Z_2^i in β .

Let $Y \in \beta'$. Write $Y = \sum_{j=1}^k a_j(x) Z_j$ for some $a_j \in C^\infty(V)$. To prove the Lemma, we need to dominate

$$\left(\int |\psi(e^{\tau Y} x) - \psi(x)|^p dx \right)^{\frac{1}{p}} = \left(\int \left| \psi \left(e^{\tau(a_1(x)Z_1 + \dots + a_k(x)Z_k)} x \right) - \psi(x) \right|^p dx \right)^{\frac{1}{p}}$$

by terms of the form

$$\left(\int |\psi(e^{sZ_j} x) - \psi(x)|^p dx \right)^{\frac{1}{p}}$$

For each $k+1 \leq i \leq n$, we will define mappings $F_i(s)(x)$ which are approximations of $e^{sZ_i} x$.

Recall that for such i , $Z_i = [Z_1^i, Z_2^i]$ where Z_1^i and Z_2^i are in β .

Define

$$F_i(s)(y) = \begin{cases} e^{-\sqrt{s}Z_2^i} e^{-\sqrt{s}Z_1^i} e^{\sqrt{s}Z_2^i} e^{\sqrt{s}Z_1^i}(y), & s \geq 0 \\ e^{-\sqrt{|s|}Z_1^i} e^{-\sqrt{|s|}Z_2^i} e^{\sqrt{|s|}Z_1^i} e^{\sqrt{|s|}Z_2^i}(y), & s < 0. \end{cases}$$

By the Campbell-Hausdorff formula,

$$F_i(s)(x) = e^{sZ_i} g(x, s)$$

where

$$g(x, s) = x + 0 \left(|s|^{3/2} \right) \quad (*)$$

and hence each F_i is C^1 . For $s = (s_1, \dots, s_n)$ and x small, define

$$F(s, x) = e^{s_1 Z_1} \dots e^{s_k Z_k} F_{k+1}(s_{k+1}) \dots F_n(s_n) x$$

The estimate in $(*)$ tells us that for each x , $F(s, x)$ is a C^1 diffeomorphism from a neighborhood of 0 in s space to a neighborhood of x . In fact, there is $\varepsilon > 0$ and neighborhoods V_1 and V_2 of 0, $V_2 \subset \subset V_1$, such that for each x in V_2 , $s \mapsto F(s, x)$ is a diffeomorphism from $B_\varepsilon(0)$ into V_1 .

It follows that for τ near 0, the implicit function theorem gives us functions

$$s(\tau, x) = (s_1(\tau, x), \dots, s_n(\tau, x))$$

such that $s(\tau, x) = o(|\tau|)$ and

$$F(s(\tau, x), x) = e^{\tau(a_1(x)Z_1 + \dots + a_k(x)Z_k)}x = e^{\tau Y}x.$$

We therefore need to dominate terms of the form

$$\left(\int \left| \psi \left(e^{s_1(\tau, x)Z_1} \dots e^{s_k(\tau, x)Z_k} \dots e^{b_1(\tau, x)X_1} \dots e^{b_\ell(\tau, x)X_\ell} x \right) - \psi(x) \right|^p dx \right)^{\frac{1}{p}},$$

where the $X_i \in \beta$, $b_j(\tau, x) = o(|\tau|)$,

$$s_i(\tau, x) = o(|\tau|).$$

After using the triangle inequality and change of variables, we are led to terms of the form

$$\left(\int \left| \psi \left(e^{b(\tau, x)Z} x \right) - \psi(x) \right|^p dx \right)^{\frac{1}{p}},$$

where $Z \in \beta$ and $b(\tau, x) = o(|\tau|)$. Finally, an application of the technique used to prove Lemma 4.1 in [4] enables us to dominate these latter terms by integrals of the form

$$\sup_{|s| \leq C|\tau|} \left(\int \left| \psi \left(e^{sZ} x \right) - \psi(x) \right|^p dx \right)^{\frac{1}{p}},$$

where C is independent of ψ . The lemma follows from these observations.

If $\beta = \{Y_1, \dots, Y_{k+1}\}$ is a basis of \mathcal{V} over a neighborhood U of 0 in \mathbb{R}^{n+1} , and $\varphi \in C_0^\infty(U)$, we define

$$\|\varphi\|_{W_{1,p}(U, \beta)} = \|\varphi\|_{L^p} + \sum_{j=1}^{k+1} \|Y_j \varphi\|_{L^p}.$$

It is clear that if $\tilde{\beta}$ is also a basis of \mathcal{V} over U , we get an equivalent norm. Hence in the sequel, we'll often omit mention of the basis.

We are now ready to state the main results of this article:

Theorem 1.1. *Let $1 < p < \infty$. Let β and β' be any bases near 0 of \mathcal{V} and $\mathcal{V} \cap T\mathbb{R}_x^n$ respectively. Then there exist neighborhoods U of 0 in \mathbb{R}^{n+1} and V of 0 in \mathbb{R}_x^n and $C > 0$ such that if $\varphi \in C_0^\infty(U)$ and $R\varphi(x) = \varphi(x, 0)$, then*

$$\|R\varphi\|_{W_{1-\frac{1}{p}, p}(V, \beta')} \leq C \|\varphi\|_{W_{1,p}(U, \beta)}.$$

Conversely, we'll prove the following extension theorem.

Theorem 1.2. *Let $1 < p < \infty$. Let β and β' be any bases near 0 of \mathcal{V} and $\mathcal{V} \cap T\mathbb{R}_x^n$ respectively. Then there exist neighborhoods U of 0 in \mathbb{R}^{n+1} and V of 0 in \mathbb{R}_x^n and a linear extension mapping from $W_{1-\frac{1}{p},p}(V, \beta')$ to $W_{1,p}(U, \beta)$ that is continuous.*

Remark 1. Theorem 1.2 shows that the loss $\frac{1}{p}$ of smoothness in Theorem 1.1 is sharp.

Remark 2. As indicated in the introduction, these theorems show that traces lose exactly the same smoothness as in the classical case.

Proof of Theorem 1.1. We begin by observing that we can choose sections X_1, \dots, X_k of \mathcal{V} of the form

$$X_i = \sum_{j=1}^n a_{ij}(x, t) \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq k$$

such that $\beta' = \{Y_1, \dots, Y_k\}$ where $Y_i = X_i|_{t=0}$

To see this, let $\beta' = \{Y_1, \dots, Y_k\}$ and choose a basis $Z = \{Z_1, \dots, Z_{k+1}\}$ of \mathcal{V} of the form

$$Z_i = \sum_{j=1}^n b_{ij}(x, t) \frac{\partial}{\partial x_j} \quad \text{for } 1 \leq i \leq k$$

and

$$Z_{k+1} = \frac{\partial}{\partial t} + \sum_{j=1}^n C_j(x, t) \frac{\partial}{\partial x_j}.$$

Such a basis Z is possible since \mathcal{V} is not contained in $T\mathbb{R}_x^n$. Let $f_{ij}(x)$ be C^∞ functions such that

$$Y_i = \sum_{j=1}^n f_{ij}(x) Z_j|_{t=0} \quad \text{for } 1 \leq i \leq k.$$

Set $X_i = \sum_{j=1}^n f_{ij}(x) Z_j$ for $1 \leq i \leq k$. Then $\{X_1, \dots, X_k\}$ is as desired. Moreover, if $X_{k+1} = Z_{k+1}$, then $\{X_1, \dots, X_{k+1}\}$ is a basis of \mathcal{V} near 0.

Next we observe that we may assume X_{k+1} to be $\frac{\partial}{\partial t}$.

Indeed, suppose $G(x, t)$ is a diffeomorphism from (x, t) space to (y, s) space such that $G(x, 0) = (x, 0)$ and $G_*(X_{k+1}) = \frac{\partial}{\partial s}$. Since $G(x, 0) = (x, 0)$, we observe that it suffices to prove the theorem in (y, s) space for the bundles $G_*(\mathcal{V})$ and $G_*(\mathcal{V}) \cap T\mathbb{R}_y^n$. Thus we will assume that $\{X_1, \dots, X_{k+1}\}$ is a basis of \mathcal{V} near 0, $X_i|_{t=0} = Y_i$ for $1 \leq i \leq k$ and $X_{k+1} = \frac{\partial}{\partial t}$.

Let $U = V \times (-\varepsilon, \varepsilon)$ be a neighborhood of 0 in \mathbb{R}^{n+1} over which the X_j span \mathcal{V} . Fix $X \in \{X_1, \dots, X_k\}$, and let $L = \frac{\partial}{\partial t} - X$. If $\varphi(x, t) \in C_0^\infty(U)$, we will express $\varphi(x, t)$ in terms of $L\varphi(x, t) = f(x, t)$ and $\varphi_0(x) = \varphi(x, 0)$ as follows.

Let $p_j(x, t)$ ($1 \leq j \leq n$) be the unique solution of

$$\begin{cases} Lp_j(x, t) &= 0 \\ p_j(x, 0) &= x_j \end{cases}$$

in a neighborhood of 0 which we still call U . Define $G(x, t) = (p(x, t), t)$ where $p = (p_1, \dots, p_n)$. Consider the change of variables

$$(x, t) \mapsto (y, s) = (p(x, t), t).$$

If $g = g(y, s)$, we have:

$$L(g(G(x, t))) = \frac{\partial g}{\partial s}(G(x, t)).$$

Hence if $H(y, s) = (h(y, s), s)$ is the inverse of G and F solves $\frac{\partial F}{\partial s}(y, s) = f(H(y, s))$, $F(y, 0) = \varphi_0(y)$, then

$$\varphi(x, t) = F(G(x, t)).$$

Hence

$$\varphi(x, t) = \varphi_0(p(x, t)) + \int_0^t f(H(p(x, t), \tau)) d\tau \quad (1.1)$$

If $\psi = \psi(x)$, let $G(t)\psi(x)$ denote the function $\psi(p(x, t))$. Using this notation we can write

$$\varphi(x, t) = G(t)\varphi_0(x) + \int_0^t f(H(p(x, t), \tau)) d\tau \quad (1.1')$$

The proof of Theorem 1.1 will use the following: (X will continue to denote an element of $\{X_1, \dots, X_k\}$.)

Lemma 1.3. *There exist neighborhoods V, U of 0 in \mathbb{R}^n and \mathbb{R}^{n+1} respectively, $\delta > 0$ and $C > 0$ such that for any $\varphi \in C_0^\infty(U)$,*

$$\left\{ \int_0^\delta \left[t^{\frac{1}{p}-1} \sup_{|\tau| \leq t} \|G(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} \right]^p \frac{dt}{t} \right\}^{\frac{1}{p}} \leq C \left(\left\| \frac{\partial \varphi}{\partial t} \right\|_{L^p(U)} + \|X\varphi\|_{L^p(U)} \right).$$

Proof of Lemma 1.3. We take $U = V \times (-\varepsilon, \varepsilon)$ so that (1.1') is valid. From (1.1') we have:

$$G(\tau)\varphi_0(x) - \varphi_0(x) = \int_0^\tau \frac{\partial \varphi}{\partial s}(x, s) ds - \int_0^\tau f(H(p(x, \tau), s)) ds.$$

Since the x support of $f(x, t) = L\varphi(x, t)$ is in V , Minkowski's inequality yields

$$\|G(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} \leq C \left\{ \int_0^\tau \left\| \frac{\partial \varphi}{\partial s}(\cdot, s) \right\|_{L^p(V)} ds + \int_0^\tau \|f(\cdot, s)\|_{L^p(V)} ds \right\}.$$

Thus for any $t \in [0, \varepsilon)$, we have:

$$t^{-1} \sup_{|\tau| \leq t} \|G(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} \leq C \left\{ t^{-1} \int_0^t \left\| \frac{\partial \varphi}{\partial s}(\cdot, s) \right\|_{L^p(V)} ds + t^{-1} \int_0^t \|L\varphi(\cdot, s)\|_{L^p(V)} ds \right\}.$$

To the latter we apply the Hardy-Littlewood inequality to get the Lemma for any $\delta \leq \varepsilon$.

(Recall that the Hardy-Littlewood inequality says that

$$\left\{ \int_0^\infty \left| t^{-1} \int_0^t h(s) ds \right|^q dt \right\}^{\frac{1}{q}} \leq C \left(\int_0^\infty |h(\tau)|^q d\tau \right)^{\frac{1}{q}}$$

for $1 < q < \infty$).

End of the proof Theorem 1.1. As indicated already, we may let

$$X_i = \sum_{j=1}^n a_{ij}(x, t) \frac{\partial}{\partial x_j} \quad \text{for } 1 \leq i \leq k,$$

$$X_i|_{t=0} = Y_i \quad \text{and} \quad X_{k+1} = \frac{\partial}{\partial t}.$$

For each $i = 1, \dots, k$, let $B_i(x, t) = e^{tY_i}x$ where we view Y_i as a vector field in \mathbb{R}^{n+1} . Let $p^i = (p_1^i, \dots, p_n^i)$ for $i \leq k$ be the unique solution of

$$\begin{cases} \frac{\partial p_j^i}{\partial t}(x, t) - X_i p_j^i &= 0 \\ p_j^i(x, 0) &= x_j. \end{cases}$$

Since $p^i(x, 0) = B_i(x, 0)$ and $X_i|_{t=0} = Y_i$, we have:

$$p^i(x, t) = B_i(x, t) + o(t^2).$$

Therefore, if $R_i(t)x$ denotes $e^{-tY_i}p^i(x, t)$, then

$$R_i(t)x = x + o(t^2) \quad \text{and} \quad R_i(t)^{-1}x = x + o(t^2).$$

Let $R^i(t) = R_i(t)^{-1}$. We have:

$$e^{tY_i}y = p^i(R^i(t)y, t) = G_i(t)(R^i(t)y)$$

where we have used the notation $G_i(t)x = p^i(x, t)$.

Let V' be a neighborhood of 0 such that $V' \subseteq V$ and

$$R^i(\tau)(V') \subseteq V \quad \forall i = 1, \dots, k \text{ and for } 0 \leq \tau \leq \delta_1, \delta_1 < \varepsilon.$$

Let

$$\omega_i(t, \varphi_0, V') = \sup_{|\tau| \leq t} \|e^{\tau Y_i} \varphi_0 - \varphi_0\|_{L^p(V')}.$$

From

$$\varphi_0(e^{\tau Y_i}x) - \varphi_0(x) = \varphi_0(G_i(\tau)(R^i(\tau)x)) - \varphi_0(R^i(\tau)x) + \varphi_0(R^i(\tau)x) - \varphi_0(x),$$

we have, for $0 \leq t \leq \delta_1$,

$$\omega_i(t, \varphi_0, V') \leq C \left\{ \sup_{|\tau| \leq t} \|R^i(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} + \sup_{|\tau| \leq t} \|G_i(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} \right\} \quad (1.2)$$

Recall that $R^i(\tau)x = x + o(\tau^2)$ and so by Lemma 3.4 in Hörmander ([4]), we get:

$$\sup_{|\tau| \leq t} \|R^i(\tau)\varphi_0 - \varphi_0\|_{L^p(V)} \leq C \omega(t^2, \varphi_0, V) \quad (1.3)$$

where

$$\omega(t^2, \varphi_0, V) = \sup_{|s| \leq t^2} \|\varphi(\cdot + s) - \varphi(\cdot)\|_{L^p(V)}$$

is the usual L^p modulus of continuity.

From the inequalities (1.2), (1.3) and Lemma 1.3 we get

$$\begin{aligned} & \left\{ \int_0^{\delta_1} \left[t^{\frac{1}{p}-1} \omega_i(t, \varphi_0, V') \right]^p \frac{dt}{t} \right\}^{\frac{1}{p}} \\ & \leq C \left\{ \left(\int_0^{\delta_1} t^{\frac{1}{p}-1} \sup_{|\tau| \leq t} \|G_i(\tau)\varphi_0 - \varphi_0\|_{L^p(V)}^p \frac{dt}{t} \right)^{\frac{1}{p}} \right. \\ & \quad \left. + \left(\int_0^{\delta_1} \left[t^{\frac{1}{p}-1} \omega(t^2, \varphi_0, V) \right]^p \frac{dt}{t} \right)^{\frac{1}{p}} \right\} \\ & \leq C \left\{ \|\varphi\|_{W_{1,p}(U)} + \left(\int_0^{\delta_1} [t^{-\sigma} \omega(t, \varphi_0, V)]^p \frac{dt}{t} \right)^{\frac{1}{p}} \right\} \end{aligned} \quad (1.4)$$

where $\sigma = \frac{1}{2} \left(1 - \frac{1}{p} \right)$.

The term $\left(\int_0^{\delta_1} [t^{-\sigma} \omega(t, \varphi_0, V)]^p dt \right)^{\frac{1}{p}}$ is the main part of the norm in the Besov space $B_p^\sigma(V)$.

We claim that for V, U small enough, $\exists C > 0$ such that

$$\|\varphi_0\|_{B_p^\sigma(V)} \leq C \|\varphi\|_{W_{1,p}(U)}. \quad (1.5)$$

Indeed, first note that since $\{X_i|_{t=0} : 1 \leq i \leq k\}$ satisfy the Hörmander condition of order 2, if the neighborhood U is small enough, the fields $\{X_i : 1 \leq i \leq k+1\}$ will satisfy the same condition in U . Hence by a result in [7] we have

$$\|\varphi\|_{L_p^{\frac{1}{2}}(\mathbb{R}^{n+1})} \leq C \|\varphi\|_{W_{1,p}(U)} \quad \text{for } \varphi \in C_0^\infty(U), \quad (1.6)$$

where $L_p^{\frac{1}{2}}(\mathbb{R}^{n+1})$ is the space of Bessel potentials in \mathbb{R}^{n+1} (see [10] for definition).

Thus

$$\left\| \frac{\partial \varphi}{\partial t} \right\|_{L^p} + \|\varphi\|_{L_p^{\frac{1}{2}}} \leq C \|\varphi\|_{W_{1,p}(U)} \quad (1.7)$$

for $\varphi \in C_0^\infty(U)$ since $X_{k+1} = \frac{\partial}{\partial t}$.

Next we recall the trace theorem (see [11])

$$\|\varphi_0\|_{B_p^\sigma(V)} \leq C \left(\left\| \frac{\partial \varphi}{\partial t} \right\|_{L^p} + \|\varphi\|_{L^{\frac{1}{p}}} \right). \quad (1.8)$$

From (1.7) and (1.8), we get

$$\|\varphi_0\|_{B_p^\sigma(V)} \leq C \|\varphi\|_{W_{1,p}(U)}.$$

The latter together with inequality (1.4) prove the theorem.

Remark 3. Since our vector fields satisfy Hörmander's bracket condition of order 2, we were able to use inequalities (1.6) and (1.8). Although the version of (1.6) for commutators of all orders is known (see [5]), we have not been able to exploit it to get a reasonable generalization of Theorem 1.1.

At this point, before we proceed to the extension theorem, we would like to stress the implications of Lemma 1.3. This lemma tells us that even when the vector field X is singular on \mathbb{R}_x^n , the restriction $\varphi_0(x) = \varphi(x, 0)$ may gain some smoothness along some direction in x space.

As an example, let $X = t^m \frac{\partial}{\partial x_1}$ where m is a positive integer. In the notation used in the lemma, we get information on the L^p modulus of

$$\varphi_0 \left(x_1 + \frac{\tau^{m+1}}{m+1}, x_2, \dots, x_n \right) - \varphi_0(x_1, x_2, \dots, x_n).$$

2. THE EXTENSION OPERATOR.

We now fix a special basis $\left\{ X_1, \dots, X_k, \frac{\partial}{\partial t} \right\}$ of \mathcal{V} of the form

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=k+1}^n a_{ij}(x, t) \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq k$$

which is achievable after a permutation of the x coordinates. The vector field $\frac{\partial}{\partial t}$ comes after using a diffeomorphism that preserves the x space as we saw before.

Let $Z_i = X_i|_{t=0}$ for $1 \leq i \leq k$. By the hypotheses on $\mathcal{V} \cap T\mathbb{R}_x^n$, the Z_i together with their brackets span $T\mathbb{R}_x^n$ near 0. We may therefore choose Z_{k+1}, \dots, Z_n such that $\{Z_1, \dots, Z_n\}$ is a basis of $T\mathbb{R}^n$ and for $i > k$ each Z_i has the form $[Z_\ell, Z_m]$ for some $\ell, m \leq k$.

For V a sufficiently small neighborhood of 0 in \mathbb{R}_x^n and t small, define

$$H_i \varphi(x, t) = \begin{cases} \frac{1}{t} \int_0^t \varphi(e^{\tau Z_i} x) d\tau, & i \leq k \\ \frac{1}{t^2} \int_0^{t^2} \varphi(e^{\tau Z_i} x) d\tau, & i \geq k+1, \end{cases}$$

where $t > 0$ and $\varphi \in C_0^\infty(V)$.

Define $H\varphi(x, t) = (H_1 \circ \cdots \circ H_n \varphi)(x, t)$, where for $\psi = \psi(x, t)$, $H_i \psi(x, t)$ is defined by letting H_i act on the function

$$x \mapsto \psi(x, t).$$

For $\varphi \in C_0^\infty(V)$, define

$$E\varphi(x, t) = \begin{cases} H\varphi(x, t), & t \in (0, \delta) \\ \varphi(x), & t = 0, \end{cases}$$

where δ is a sufficiently small positive number.

Let S be the Seeley extension operator (see [8]) from $C^\infty(\overline{\mathbb{R}_+^n})$ to $C^\infty(\mathbb{R}^{n+1})$.

Let $\rho_\delta \in C_0^\infty(-\delta, \delta)$ such that $\rho_\delta(0) = 1$.

Because of Lemma 1.1, Theorem 1.2 will follow from the following:

Proposition 2.1. *Let $\beta = \{Z_1, \dots, Z_k\}$. If V and V' are small enough, $V \subset \subset V'$, there exist U a neighborhood of 0 in \mathbb{R}^{n+1} , $C > 0$ and $\delta > 0$ such that for any $\psi \in C_0^\infty(V)$,*

$$\|\rho_\delta S(E\psi)\|_{W_{1,p}(U)} \leq C \|\psi\|_{W_{1-\frac{1}{p},p}(V, V', \beta)}.$$

(Here $\beta = \{Z_1, \dots, Z_n\}$ is the special basis chosen in this section).

The proof of this proposition will be based on some lemmas. For $\tau = (\tau_1, \dots, \tau_n)$ and $x \in V$, let

$$\eta(\tau, x) = e^{\tau_1 Z_1} \circ \cdots \circ e^{\tau_n Z_n} x$$

where $\tau = (\tau_1, \dots, \tau_n)$.

If the neighborhood V of 0 in \mathbb{R}_x^n is sufficiently small, $\eta(\tau, x)$ is a diffeomorphism from a neighborhood of 0 in τ space into V .

Lemma 2.2. *In the coordinates of $\eta(\tau, x)$, we have*

$$Z_j = \frac{\partial}{\partial \tau_j} + \sum_{\ell=k+1}^n \varsigma_{\ell j}(\tau, x) \frac{\partial}{\partial \tau_\ell}$$

where each $\varsigma_{\ell j}(\tau, x) = o(\tau)$ for each $1 \leq j \leq k$;

$$Z_i = \sum_{\ell=k+1}^n \varsigma_{i\ell}(\tau, x) \frac{\partial}{\partial \tau_\ell} \quad \text{for } k+1 \leq i \leq n.$$

Proof. Recall that

$$Z_i = \frac{\partial}{\partial x_i} + \sum_{j=k+1}^n a_{ij}(x) \frac{\partial}{\partial x_j}$$

($a_{ij}(x) = a_{ij}(x, 0)$) for $1 \leq i \leq k$. Moreover, for each $\ell \geq k+1$, $\exists i, j$ in $\{1, \dots, k\}$ such that $Z_\ell = [Z_i, Z_j]$.

Therefore, for $\ell \geq k+1$, each

$$Z_\ell = \sum_{j=k+1}^n b_{\ell j}(x) \frac{\partial}{\partial x_j}$$

for some smooth $b_{\ell j}$.

It follows that

$$\eta(\tau, x) = (x_1 + \tau_1, \dots, x_k + \tau_k, x_{k+1} + \tau \cdot g^{k+1}(\tau, x), \dots, x_n + \tau \cdot g^n(\tau, x))$$

where for $i \geq k+1$, $\tau \cdot g^i(\tau, x) = \sum_{\ell=1}^n \tau_\ell g_\ell^i(\tau, x)$ for some C^∞ functions $g_\ell^i(\tau, x)$.

Moreover, the diffeomorphism $\tau \mapsto \eta(\tau, x)$ maps each $\frac{\partial}{\partial \tau_j} \Big|_0$ to $Z_j|_x$ for $1 \leq j \leq k$. The lemma follows from these remarks.

Lemma 2.3. *Let B_1, \dots, B_k be any operators. Then*

- (a) $B_1 B_2 \cdots B_k - I = (B_1 - I)B_2 \cdots B_k + (B_2 - I)B_3 \cdots B_k + \cdots + (B_k - I);$
- (b) *The product $B_1 \cdots B_{k-1} (B_k - I)$ is a linear combination of terms of the form*

$$(B_{i_1} - I)(B_{i_2} - I) \cdots (B_{i_j} - I), \quad 1 \leq j \leq k$$

and $1 \leq i_1 < \cdots < i_j \leq k$.

Note that (b) can easily be proved by induction and (a) is obvious. We remark that (a) was used in [4].

Proof of Proposition 2.1. Recall that for $\psi \in C_0^\infty(V)$ and $0 < t \leq \delta$,

$$H\psi(x, t) = H_1 \circ \cdots \circ H_n \psi(x, t).$$

We will estimate $\|\partial_t(H\psi)\|_{L^p(V \times (-\delta, \delta))}$ and

$$\|X_i(H\psi)\|_{L^p(V \times (-\delta, \delta))}.$$

Observe that $\frac{\partial}{\partial t}(H\psi)$ is a sum of terms of the form

$$H_1 \circ \cdots \circ \partial_t H_i \circ \cdots \circ H_n \psi,$$

where for $1 \leq i \leq k$,

$$\partial_t H_i f(x, t) = \frac{-1}{t^2} \int_0^t (e^{\tau Z_i} - I) f(x) d\tau + \frac{1}{t} (e^{t Z_i} - I) f(x) \quad (2.1)$$

while for $k+1 \leq i \leq n$,

$$\partial_t H_i f(x, t) = \frac{-2}{t^3} \int_0^{t^2} (e^{\tau Z_i} - I) f(x) d\tau + \frac{2}{t} (e^{t^2 Z_i} - I) f(x) \quad (2.2)$$

Writing each H_j as $(H_j - I) + I$, we can express $H_1 \circ \cdots \circ \partial_t H_i \circ \cdots \circ H_n \psi$ as a sum of terms of the form:

$$(H_{i_1} - I) \circ \cdots \circ \partial_t H_i \circ \cdots \circ (H_{i_m} - I) \psi(x, t).$$

Now if $i_1 \leq k$, the latter can be bounded by a sum of terms of the form

$$\frac{1}{t} \cdot |\psi(e^{\tau_{i_1} Z_{i_1}} y) - \psi(y)| \quad (2.3)$$

where $y = e^{s_{j_1} Z_{j_1}} \cdots e^{s_{j_\ell} Z_{j_\ell}} x$ for some $s_{j_1}, \dots, s_{j_\ell} \in [0, t]$.

If $i_1 > k$, we use

$$\frac{1}{t} \cdot |\psi(e^{\tau_{i_1}^2 Z_{i_1}} y) - \psi(y)|, \quad |\tau_{i_1}| \leq t. \quad (2.4)$$

Let $V_1 \subset\subset V$ be a neighborhood of 0 such that

$$e^{\tau_{i_1} Z_{i_1}} \cdots e^{\tau_{i_m} Z_{i_m}} (V_1) \subseteq V \quad \text{for } |\tau_{i_j}| \leq \delta,$$

and $1 \leq i_1 < \cdots < i_m \leq n$.

From (2.1) – (2.4), we get

$$\begin{aligned} & \left\| \frac{\partial(H\psi)}{\partial t} \right\|_{L^p(V_1 \times (0, \delta))} \\ & \leq C \left\{ \sum_{i=1}^k \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_i(t, \psi, V) \right)^p \frac{dt}{t} \right]^{\frac{1}{p}} + \sum_{j=k+1}^n \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_j(t^2, \psi, V) \right)^p \frac{dt}{t} \right]^{\frac{1}{p}} \right\}. \end{aligned} \quad (2.5)$$

Next for $j \geq k+1$, we estimate $\omega_j(t^2, \psi, V)$ which by definition

$$= \sup_{|\tau| \leq t^2} \left\| \psi(e^{\tau^2 Z_j} x) - \psi(x) \right\|_{L^p(V)}.$$

Since $j \geq k+1$, $\exists m, \ell \leq k$ such that $Z_j = [Z_m, Z_\ell]$.

We have:

$$\begin{aligned} \left| \psi(e^{\tau^2 Z_j} x) - \psi(x) \right| & \leq \left| \psi(e^{\tau^2 Z_j} x) - \psi(e^{-\tau Z_m} e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x) \right| \\ & \quad + \left| \psi(e^{-\tau Z_m} e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x) - \psi(e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x) \right| \\ & \quad + \left| \psi(e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x) - \psi(e^{\tau Z_m} e^{\tau Z_\ell} x) \right| \\ & \quad + \left| \psi(e^{\tau Z_m} e^{\tau Z_\ell} x) - \psi(e^{\tau Z_\ell} x) \right| \\ & \quad + \left| \psi(e^{\tau Z_\ell} x) - \psi(x) \right|. \end{aligned} \quad (2.6)$$

In the sum on the right in (2.6), every term except the first one can be estimated by

$$|\psi(e^{-\tau Z_m} y) - \psi(y)| + |\psi(e^{-\tau Z_\ell} y) - \psi(y)|$$

where y varies in V provided $x \in V_1$.

To estimate the L^p norm of the first term, we recall first from the Campbell-Hausdorff formula that

$$\left| e^{\tau^2 Z_j} x - e^{-\tau Z_m} e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x \right| = o(\tau^3),$$

as long as x varies in the relatively compact set V_1 .

The latter allows us to apply Lemma 3.4 of [4] to conclude that

$$\begin{aligned} \left(\int_{V_1} \left| \psi(e^{\tau^2 Z_j} x) - \psi(e^{-\tau Z_m} e^{-\tau Z_\ell} e^{\tau Z_m} e^{\tau Z_\ell} x) \right|^p dx \right)^{\frac{1}{p}} &\leq C \sup_{|s| \leq t^3} \|\psi(\cdot + s) - \psi(\cdot)\|_{L^p(V)} \\ &= C \omega(t^3, \psi, V), \end{aligned}$$

where ω is the usual modulus of L^p continuity.

This inequality together with (2.6) imply that when $j \geq k+1$,

$$\omega_j(t^2, \psi, V) \leq C \left(\sum_{i=1}^k \omega_i(t, \psi, V) + \omega(t^3, \psi, V) \right) \quad (2.7)$$

Next observe that

$$\begin{aligned} \left(\int_0^\delta \left[t^{\frac{1}{p}-1} \omega(t^3, \psi, V) \right]^p \frac{dt}{t} \right)^{\frac{1}{p}} &= \left(\int_0^\delta \left(\frac{\omega(s, \psi, V)}{s^{1/3(1-\frac{1}{p})}} \right)^p \frac{ds}{s} \right)^{\frac{1}{p}} \\ &\leq \delta^{1/6(1-\frac{1}{p})} \left(\int_0^\delta \left(\frac{\omega(s, \psi, V)}{s^{1/2(1-\frac{1}{p})}} \right)^p \frac{ds}{s} \right)^{\frac{1}{p}} \\ &\leq C_1 \delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V \times (0, \delta))} \quad (\text{by (1.5)}). \end{aligned} \quad (2.8)$$

From (2.5), (2.7) and (2.8) we conclude:

$$\begin{aligned} \|\partial_t(H\psi)\|_{L^p(V_1 \times (0, \delta))} &\leq C \left\{ \sum_{i=1}^k \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_i(t, \psi, V) \right)^p \frac{dt}{t} \right]^{\frac{1}{p}} \right. \\ &\quad \left. + \delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V \times (0, \delta))} \right\}. \end{aligned} \quad (2.8')$$

We next estimate $\|X_i(H\psi)\|_{L^p(V \times (0, \delta))}$ for $i = 1, \dots, k$.

Recall that $\eta(\tau, x) = e^{\tau_1 Z_1} \dots e^{\tau_n Z_n} x$ and

$$H\psi(x, t) = \frac{1}{t^{2n-k}} \int_0^t \dots \int_0^{t^2} \psi(\eta(\tau, x)) d\tau \quad (d\tau = d\tau_1 \dots d\tau_n).$$

We had

$$\eta_i(\tau, x) = \begin{cases} x_i + \tau_i, & i \leq k \\ x_i + \tau \cdot g^i(x, t), & i \geq k+1, \end{cases}$$

$$X_j|_{t=0} = Z_j = \frac{\partial}{\partial x_j} + \sum_{\ell=k+1}^n a_{j\ell}(x) \frac{\partial}{\partial x_\ell}.$$

To estimate $X_j(H\psi)$, we will first compute $Z_j(H\psi)$. We will need to compare

$$Z_j\{\psi(\eta(\tau, x))\} \quad \text{with} \quad (Z_j\psi)(\eta(\tau, x)).$$

From the expressions of the Z_j and the η_i , we get:

$$Z_j\{\psi(\eta(\tau, x))\} = (Z_j\psi)(\eta(\tau, x)) + \sum_{\ell=k+1}^n p_{j\ell}(x, \tau) \frac{\partial \psi}{\partial x_\ell}(\eta(\tau, x)),$$

where $p_{j\ell}(x, \tau) = o(\tau)$, $1 \leq j \leq k$.

Observe that from the form of $\{Z_1, \dots, Z_n\}$, each $\frac{\partial}{\partial x_\ell}$ for $\ell \geq k+1$ is a linear combination of Z_{k+1}, \dots, Z_n . Therefore, we get:

$$Z_j\{\psi(\eta(\tau, x))\} = (Z_j\psi)(\eta(\tau, x)) + \sum_{\ell=k+1}^n q_{j\ell}(x, \tau) (Z_\ell\psi)(\eta(\tau, x)), \quad (2.9)$$

$q_{j\ell}(x, \tau) = o(\tau)$ and $1 \leq j \leq k$.

By similar arguments, we can get a relation of the type (2.9) for Z_i when $i \geq k+1$.

Now, for $1 \leq i \leq k$,

$$X_i = Z_i + t \sum_{j=k+1}^n C_{ij}(x, t) Z_j \quad \text{for some } C^\infty C_{ij}.$$

Hence by (2.9) and its analogue for $j \geq k+1$,

$$\begin{aligned} X_i\{\psi(\eta(\tau, x))\} &= (Z_i\psi)(\eta(\tau, x)) + t \sum_{j=k+1}^n r_{ij}(x, t, \tau) (Z_j\psi)(\eta(\tau, x)), \\ &\text{for } 1 \leq i \leq k, \quad \text{for some } C^\infty r_{ij}. \end{aligned} \quad (2.10)$$

Using (2.10), we can write (for $1 \leq i \leq k$)

$$X_i\{H\psi(x, t)\} = \text{I} + \text{II},$$

where

$$\text{I} = \frac{1}{t^{2n-k}} \int_0^t \cdots \int_0^{t^2} (Z_i\psi)(\eta(\tau, x)) d\tau$$

and

$$\text{II} = \sum_{j=k+1}^n \frac{1}{t^{2n-k-1}} \int_0^t \cdots \int_0^{t^2} r_{ij}(x, t, \tau) (Z_j\psi)(\eta(\tau, x)) d\tau.$$

Below we will use the notations

$$\tau_i(a) = (\tau_1, \dots, \tau_{i-1}, a, \tau_{i+1}, \dots, \tau_n),$$

$\Delta_i(s)f(\tau) = f(\tau_i(s)) - f(\tau_i(0))$, and the easily verifiable identity

$$\Delta_i(s) \{g(\tau)f(\tau)\} = (g(\tau_i(s)) \cdot \text{Delta}_i(s)f(\tau)) + (f(\tau_i(0)) \cdot \Delta_i(s)g(\tau)).$$

Now by Lemma 2.2, the term

$$\begin{aligned} \text{I} &= \frac{1}{t^{2n-k}} \int_0^t \cdots \int_0^{t^2} \left[\frac{\partial}{\partial \tau_i} + \sum_{\ell=k+1}^n \varsigma_{i\ell}(\tau, x) \frac{\partial}{\partial \tau_\ell} \right] \psi(\eta(\tau, x)) d\tau \\ &= \frac{1}{t^{2n-k}} \int_0^t \cdots \int_0^t \cdots \int_0^{t^2} \Delta_i(t) \psi(\eta(\tau, x)) d\tau_1 \cdots \widehat{d\tau_i} \cdots d\tau_n \\ &\quad + \frac{1}{t^{2n-k}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \cdots \int_0^{t^2} \Delta_\ell(t^2) [\varsigma_{i\ell}(\tau, x) \psi(\eta(\tau, x))] d\tau_1 \cdots \widehat{d\tau_\ell} \cdots d\tau_n \\ &\quad - \frac{1}{t^{2n-k}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \frac{\partial \varsigma_{i\ell}}{\partial \tau_\ell}(\tau, x) \psi(\eta(\tau, x)) d\tau \\ &= \frac{1}{t^{2n-k}} \int_0^t \cdots \int_0^t \cdots \int_0^{t^2} \Delta_i(t) \psi(\eta(\tau, x)) d\tau_1 \cdots \widehat{d\tau_i} \cdots d\tau_n \\ &\quad + \frac{1}{t^{2n-k}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \cdots \int_0^{t^2} \varsigma_{i\ell}(\tau_\ell(t^2), x) \Delta_\ell(t^2) \psi(\eta(\tau, x)) d\tau_1 \cdots \widehat{d\tau_\ell} \cdots d\tau_n \\ &\quad + \frac{1}{t^{2n-k}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \cdots \int_0^{t^2} \Delta_\ell(t^2) \varsigma_{i\ell}(\tau, x) \psi(\eta(\tau_\ell(0), x)) d\tau_1 \cdots \widehat{d\tau_\ell} \cdots d\tau_n \\ &\quad - \frac{1}{t^{2n-k}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \frac{\partial \varsigma_{i\ell}}{\partial \tau_\ell}(\tau, x) \psi(\eta(\tau, x)) d\tau. \end{aligned}$$

Again by Lemma 2.2, a typical term in II is, for some $j \geq k+1$:

$$\begin{aligned} &= \frac{1}{t^{2n-k-1}} \int_0^t \cdots \int_0^{t^2} r_{ij}(x, t, \tau) (Z_j \psi)(\eta(\tau, x)) d\tau \\ &= \frac{1}{t^{2n-k-1}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} r_{ij} \varsigma_{i\ell}(\tau, x) \frac{\partial}{\partial \tau_\ell} \{ \psi(\eta(\tau, x)) \} d\tau \\ &= \frac{1}{t^{2n-k-1}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \cdots \int_0^{t^2} r_{ij}(x, t, \tau_\ell(\tau^2)) \varsigma_{i\ell}(\tau_\ell(t^2), x) \Delta_\ell(t^2) \psi(\eta) d\tau_1 \cdots \widehat{d\tau_\ell} \cdots d\tau_n \\ &\quad + \frac{1}{t^{2n-k-1}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \cdots \int_0^{t^2} \Delta_\ell(t^2) \{ r_{ij}(x, t, \tau) \varsigma_{i\ell}(\tau, x) \} \psi(\eta(\tau_\ell(0), x)) d\tau_1 \cdots \widehat{d\tau_\ell} \cdots d\tau_n \\ &\quad - \frac{1}{t^{2n-k-1}} \sum_{\ell=k+1}^n \int_0^t \cdots \int_0^{t^2} \left\{ \frac{\partial}{\partial \tau_\ell} (r_{ij} \cdot \varsigma_{i\ell}) \right\} \psi(\eta(\tau, x)) d\tau. \end{aligned}$$

By the mean value theorem,

$$\Delta_\ell(t^2)\varsigma_{i\ell}(\tau, x) = o(t^2) = \Delta_\ell(t^2)\{r_{ij}(x, t, \tau)\varsigma_{i\ell}(\tau, x)\},$$

and recall that for $i \leq k$, $\varsigma_{i\ell}(\tau, x) = o(\tau)$.

Let $P_t = \{(\tau_1, \dots, \tau_n) : |\tau_1| \leq t, \dots, |\tau_k| \leq t \text{ and } |\tau_i| \leq t^2 \text{ for } i > k\}$.

Using the estimates on I and II and the Minkowski inequality we get:

$$\begin{aligned} & \|X_i(H\psi(x, t))\|_{L^p(V \times (0, \delta))} \\ & \leq C \left[\sum_{i=1}^k \left(\int_0^\delta \left(t^{-1} \sup_{\tau \in P_t} \|\Delta_i(t)\psi(\eta(\tau)\cdot)\|_{L^p(V)} \right)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad + \sum_{j=k+1}^n \left(\int_0^\delta \left(t^{-1} \sup_{\tau \in P_t} \|\Delta_j(t^2)\psi(\eta(\tau)\cdot)\|_{L^p(V)} \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \left. + \|\psi\|_{L^p(V)} \right] \\ & = C (J_1(V) + J_2(V) + \|\psi\|_{L^p(V)}). \end{aligned} \tag{2.11}$$

To estimate $J_1(V)$ observe that by using Lemma (2.3) (b) we have:

$$\begin{aligned} & \sup_{\tau \in P_t} \|\Delta_i(t)\psi(\eta(\tau)\cdot)\|_{L^p(V)} \\ & = \sup_{\tau \in P_t} \|e^{\tau_1 Z_1} \dots e^{\tau_{i-1} Z_{i-1}} (e^{\tau_i Z_i} - I) e^{\tau_{i+1} Z_{i+1}} \dots e^{\tau_n Z_n} \psi(\cdot)\|_{L^p(V)} \\ & \leq C \sup_{\tau \in P_t} \|e^{\tau_1 Z_1} \dots e^{\tau_{i-1} Z_{i-1}} (e^{\tau_i Z_i} - I) \psi(\cdot)\|_{L^p(V')} \\ & \leq C \sum_{i=1}^k \omega_i(t, \psi, V'). \end{aligned} \tag{2.12}$$

where V' is a small neighborhood of V . Indeed, by decreasing δ , we can make V' as close to V as we wish. It follows that

$$J_1(V) \leq C \sum_{i=1}^k \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_i(t, \psi, V') \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}. \tag{2.13}$$

The term $J_2(V)$ can be estimated by using the arguments employed to establish (2.7) and (2.8).

This yields:

$$\begin{aligned} J_2(V) & \leq C \left\{ \sum_{i=1}^k \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_i(t, \psi, V') \right)^p \frac{dt}{t} \right]^{\frac{1}{p}} \right. \\ & \quad \left. + \delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V' \times (0, \delta))} \right\}. \end{aligned} \tag{2.14}$$

From (2.12), (2.13) and (2.14) we get the following: if $V_1 \subset\subset V$, then $\delta > 0$ can be chosen small enough so that

$$\begin{aligned} \|X_i(H\psi)\|_{L^p(V \times (0, \delta))} &\leq C \left\{ \sum_{i=1}^k \left[\int_0^\delta \left(t^{\frac{1}{p}-1} \omega_i(t, \psi, V) \right)^p \frac{dt}{t} \right]^{\frac{1}{p}} \right. \\ &\quad \left. + \|\psi\|_{L^p(V)} + \delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V \times (0, \delta))} \right\}. \end{aligned} \quad (2.14')$$

Observe next that using Minkowski's inequality for integrals one easily gets:

$$\|H\psi\|_{L^p(V_1 \times (0, \delta))} \leq \delta^{\frac{1}{p}} \|\psi\|_{L^p(V)}. \quad (2.15)$$

From (2.8'), (2.14) and (2.15) we get the following: given $V_1 \subset\subset V$ neighborhoods of 0, there exist $\delta > 0$ and $C > 0$ such that

$$\|H\psi\|_{W_{1,p}(V_1 \times (0, \delta))} \leq C \left(\|\psi\|_{W_{1-\frac{1}{p},p}(\beta, V)} + \delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V \times (0, \delta))} \right). \quad (2.16)$$

Let now V_2 be a neighborhood of 0 such that

$$e^{\tau_1 Z_1} \dots e^{\tau_n Z_n}(V_2) \subseteq V_1$$

for $|\tau_j| \leq \delta$.

If $\psi \in C_0^\infty(V_2)$, then the x -support of $H\psi$ is in V_1 .

Therefore, for $\psi \in C_0^\infty(V_2)$, the term

$$\delta^{1/6(1-\frac{1}{p})} \|H\psi\|_{W_{1,p}(V \times (0, \delta))}$$

in (2.16) can be absorbed on the left hand side yielding: for $\psi \in C_0^\infty(V_2)$,

$$\|H\psi\|_{W_{1,p}(V_1 \times (0, \delta))} \leq C \|\psi\|_{W_{1-\frac{1}{p},p}(\beta, V_1)}. \quad (2.17)$$

The assertion in Proposition 2.1 easily follows from (2.17).

As indicated before, Proposition 2.1 and Lemma 1.1 imply Theorem 1.2.

REFERENCES

1. N. Aronszajn, *On coercive integro-differential forms*, Conference on PDE, University of Kansas (1955), 94-106.
2. O. Besov, *Investigation of a family of function spaces in connection with theorems of imbeddings and extensions*, Trudy Mat. Inst. Steklov **60** (1961), 42-81; Transl.: Amer. Math. Soc. Translations, Ser.2 **40** (1964), 161-207.
3. E. Gagliardo, *Caratheizzazione delle tracce sulla frontiera relativ ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova **27** (1957), 284-305; (M.R. 21, 1525).
4. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147-171.
5. I. Pesenson, *The trace problem and the Hardy operator for non-isotropic function spaces on the Heisenberg group*, Commun. in PDE **19** (1994), 665-676.

6. I. Pesenson, *Functions which are smooth along vector fields*, Mat. Zametki **48** (1990), 95-104; Transl.: Notes of the Academy of Sciences of the USSR **48** (1990), 683-688.
7. L. Rothschild and E. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta. Math **137** (1976), 247-320.
8. R. Seeley, *Extension of C^∞ functions defined in half-space*, Proc AMS **15** (1964), 625-626.
9. L. Slobodeckii, *Sobolev's spaces of fractional order and their application to boundary problems for pde*, Dokl. Akad. Nauk SSSR **118** (1958), 243-246.
10. E. Stein, *The characterization of functions arising as potentials, I*, Bull. AMS **67** (1961), 102-104; II **68** (1962), 577-582.
11. H. Triebel, *Interpolation theory, Function Spaces and Differential operators*, Springer-Verlag, Berlin, 1978.

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122, U.S.A.

E-mail address: `berhanu@euclid.math.temple.edu`

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122, U.S.A.

E-mail address: `pesenson@euclid.math.temple.edu`